KU LEUVEN

Extension Field Cancellation

A New MQ Trapdoor Construction

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Alan Szepieniec[1], Jintai Ding[2], Bart Preneel[1]

1: KU Leuven, ESAT/COSIC

first.secondname@esat.kuleuven.be

2: University of Cincinnati, jintai.ding@uc.edu

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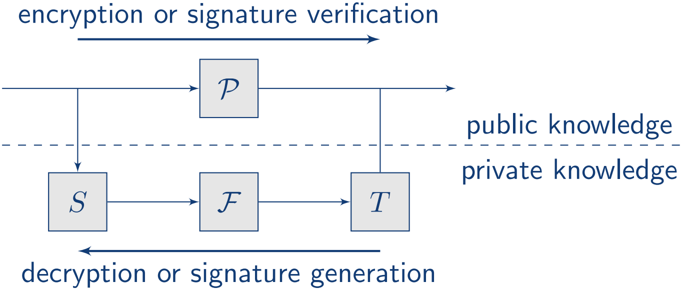
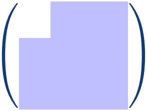
Multivariate Quadratic Cryptosystems

- public key: $\mathcal{P} \in (\mathbb{F}_q[x_1, \ldots, x_n])^m$
- public operation: evaluate in $\mathbf{x} \in \mathbb{F}_q^n$
- secret key: $(S, T, \mathcal{F})$ where
  $S \in \mathsf{GL}_n(\mathbb{F}_q), T \in \mathsf{GL}_m(\mathbb{F}_q), \mathcal{F} \in (\mathbb{F}_q[x_1, \ldots, x_n])^m$
  such that $\mathcal{P} = T \circ \mathcal{F} \circ S$
- private operation: invert $S, \mathcal{F}, T$ — all easy!



Single-Field Schemes

- all arithmetic occurs in $\mathbb{F}_q$
- canonical example: UOV
- $\mathcal{F}_i(\mathbf{o}, \mathbf{v}) = \begin{pmatrix} \mathbf{o}^\mathsf{T} & \mathbf{v}^\mathsf{T} \end{pmatrix} \mathfrak{F}_i \begin{pmatrix} \mathbf{o} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{o}^\mathsf{T} & \mathbf{v}^\mathsf{T} \end{pmatrix}$  $\begin{pmatrix} \mathbf{o} \\ \mathbf{v} \end{pmatrix}$
- invert $\mathcal{F}(\mathbf{o}, \mathbf{v}) = \mathbf{y}$:
  - fix $\mathbf{v}$ at random
  - solve $\mathcal{F}(\mathbf{o}, \mathbf{v}) = \mathbf{y}$ for $\mathbf{o}$
  - linear system!

Mixed-Field Schemes

- arithmetic occurs in $\mathbb{F}_q$ as well as in $\mathbb{F}_{q^n} \cong \mathbb{F}_q[z]/\langle p(z)\rangle$
- canonical example: HFE
- let $\varphi(\mathbf{x}) : \mathbb{F}_q^n \to \mathbb{F}_{q^n} : \mathbf{x} \mapsto \mathcal{X} = x_0 + x_1 z + \ldots x_{n-1} z^{n-1}$
- let $f(\mathcal{X}) = \sum_{i<d} \sum_{j<d} \alpha_{i,j} \mathcal{X}^{q^i + q^j} + \sum_{k<d} \beta_k \mathcal{X}^{q^k} + \gamma$
- $\mathcal{F}(\mathbf{x}) = \varphi^{-1} \circ f \circ \varphi(\mathbf{x})$
- or for simplicity: $\mathcal{F}(\mathcal{X}) = f(\mathcal{X})$
- invert $\mathcal{F}(\mathcal{X}) = \mathcal{Y}$:
    - factorize the polynomial $\mathcal{F}(\mathcal{X}) - \mathcal{Y}$
    - choose a root $\mathcal{X}_r$ such that $\mathcal{F}(\mathcal{X}_r) - \mathcal{Y} = 0$

MQ Encryption Schemes

- ZHFE
  - mixed-field
  - 2 high-degree polynomials $\mathcal{F}(\mathcal{X})$ and $\hat{\mathcal{F}}(\mathcal{X})$ linked to 1 low-degree polynomial $\Psi(\mathcal{X})$
  - inversion: factorize $\Psi(\mathcal{X})$
- ABC / Simple Matrix Encryption
  - single-field, but embeds matrix algebra
  - reduces inversion to linear system solving
- Extension Field Cancellation (EFC)
  - mixed-field
  - 2 high-degree polynomials
  - reduces inversion to linear system solving

!! All three are expanding maps $\mathbb{F}_q^n \to \mathbb{F}_q^{2n}$!!

EFC: Basic Trapdoor

- let $\varphi_m : \mathbb{F}_q^n \to \mathbb{F}_q^{n \times n}$ map a vector $\mathbf{x} \in \mathbb{F}_q^n$ to the matrix representation of $\mathcal{X} \in \mathbb{F}_{q^n}$.
- let $A, B \in \mathbb{F}_q^{n \times n}$ be matrices and $\alpha(\mathcal{X}) = \varphi(A\mathbf{x}), \quad \beta(\mathcal{X}) = \varphi(B\mathbf{x})$
- Central map:

$$\mathcal{F} = \begin{pmatrix} \varphi_m(A\mathbf{x})\mathbf{x} \\ \varphi_m(B\mathbf{x})\mathbf{x} \end{pmatrix} = \begin{pmatrix} \alpha(\mathcal{X})\mathcal{X} \\ \beta(\mathcal{X})\mathcal{X} \end{pmatrix}$$

EFC: Basic Trapdoor

Central map:

$$\mathcal{F} = \begin{pmatrix} \varphi_m(A\mathbf{x})\mathbf{x} \\ \varphi_m(B\mathbf{x})\mathbf{x} \end{pmatrix} = \begin{pmatrix} \alpha(\mathcal{X})\mathcal{X} \\ \beta(\mathcal{X})\mathcal{X} \end{pmatrix}$$

How to invert?

$$\mathcal{F}(\mathcal{X}) = \begin{pmatrix} \alpha(\mathcal{X})\mathcal{X} \\ \beta(\mathcal{X})\mathcal{X} \end{pmatrix} = \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{pmatrix}$$

Solution:

$$\beta(\mathcal{X})\mathcal{D}_1 - \alpha(\mathcal{X})\mathcal{D}_2 = 0$$

i.e., solve for $\mathbf{x}$:

$$\varphi_m(B\mathbf{x})\mathbf{d}_1 - \varphi_m(A\mathbf{x})\mathbf{d}_2 = 0$$

which is a *linear* system.

Enhanced Trapdoor

- key idea: use Frobenius isomorphism
- disadvantage: restricted to characteristic 2 only

$$\mathcal{E}(\mathcal{X}) = \begin{pmatrix} \alpha(\mathcal{X})\mathcal{X} + \beta(\mathcal{X})^3 \\ \beta(\mathcal{X})\mathcal{X} + \alpha(\mathcal{X})^3 \end{pmatrix}$$

Enhanced Trapdoor: Inversion

How to invert?

$$\mathcal{E}(\mathcal{X}) = \begin{pmatrix} \alpha(\mathcal{X})\mathcal{X} + \beta(\mathcal{X})^3 \\ \beta(\mathcal{X})\mathcal{X} + \alpha(\mathcal{X})^3 \end{pmatrix} = \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{pmatrix}$$

Solution: solve for $\mathcal{X}$:

$$\alpha(\mathcal{X})\mathcal{D}_2 - \beta(\mathcal{X})\mathcal{D}_1 = \alpha(\mathcal{X})^4 - \beta(\mathcal{X})^4$$

or for $\mathbf{x}$:

$$\alpha_m(\mathbf{x})\mathbf{d}_2 - \beta_m(\mathbf{x})\mathbf{d}_1 = Q_2(A\mathbf{x} - B\mathbf{x})$$

where $Q_2 \in \mathbb{F}_q^{n\times n}$ is the matrix associated with the Frobenius transform $\mathcal{X} \mapsto \mathcal{X}^4$.

Bilinear Attack

- basic variant: $\mathcal{F}(\mathcal{X}) = \begin{pmatrix} \alpha(\mathcal{X})\mathcal{X} \\ \beta(\mathcal{X})\mathcal{X} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{pmatrix}$
- bilinear relation: $\beta(\mathcal{X})\mathcal{Y}_1 = \alpha(\mathcal{X})\mathcal{Y}_2$
- there exists coefficients $K_i, L_i \in \mathbb{F}_{q^n}$ such that
  $$\sum_{i=0}^{n-1} \mathcal{X}^{q^i}(K_i\mathcal{Y}_1 + L_i\mathcal{Y}_2) = 0$$
- attack:
  - generate many tuples $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2)$
  - compute K_i and L_i using linear algebra
  - given a ciphertext $\boldsymbol{\mathcal{Y}} = (\mathcal{Y}_1, \mathcal{Y}_2)$ and given the coefficients K_i, L_i, computing $\mathcal{X}$ is easy

Other Attacks and Defenses

- same basic idea
- protect against Bilinear Attack: minus
- protect against Algebraic Attack: more minus
- protect against Differential Symmetry Attack: projection
- EFC_p^-, $\mathsf{EFC}_{pt^2}^-$

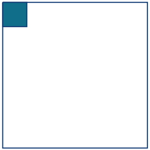
Algebraic Attack

- Algebraic Attack: decent Gröbner bases algorithms (*e.g.* F_4, F_5, MutantXL)
- Running time depends on *degree of regularity*
- D_{reg} depends on rank of quadratic form

$$\mathcal{F}(\mathcal{X}) = \begin{pmatrix} \boldsymbol{\mathcal{X}}^{\mathsf{T}} \mathfrak{F}_1 \boldsymbol{\mathcal{X}} \\ \boldsymbol{\mathcal{X}}^{\mathsf{T}} \mathfrak{F}_2 \boldsymbol{\mathcal{X}} \end{pmatrix} \quad \text{where e.g.} \quad \boldsymbol{\mathcal{X}}^{\mathsf{T}} = (\mathcal{X}, \mathcal{X}^q, \mathcal{X}^{q^2} \ldots \mathcal{X}^{q^{n-1}})$$

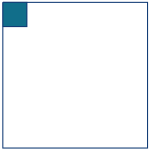
Rank of Extension Field Quadratic Form

$\mathcal{F}_1 = \alpha(\mathcal{X})\mathcal{X} \sim$



rank = 2

$\mathcal{F} \circ S \sim$



rank = 2

(change of basis)

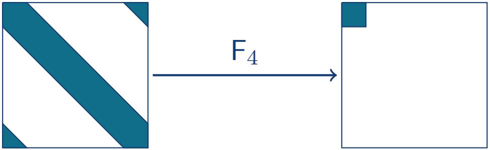
$T \circ \mathcal{F} \circ S \sim$

full rank

$$T(\mathcal{X}) = \sum t_i \mathcal{X}^{q^i}$$

$$T \circ \mathcal{F}(\mathcal{X}) = \sum t_i \left(\boldsymbol{\mathcal{X}}^{\mathsf{T}} \mathfrak{F} \boldsymbol{\mathcal{X}}\right)^{q^i}$$

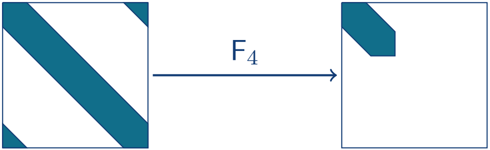
Fast Gröbner Basis



- F_4 implicitly recovers T

Minus

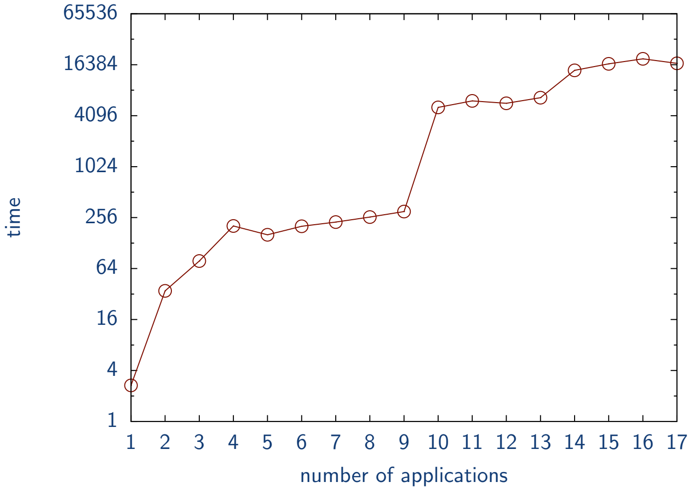
- solution: drop a rows from T
- F_4 can only recover $n-a$ rows of T



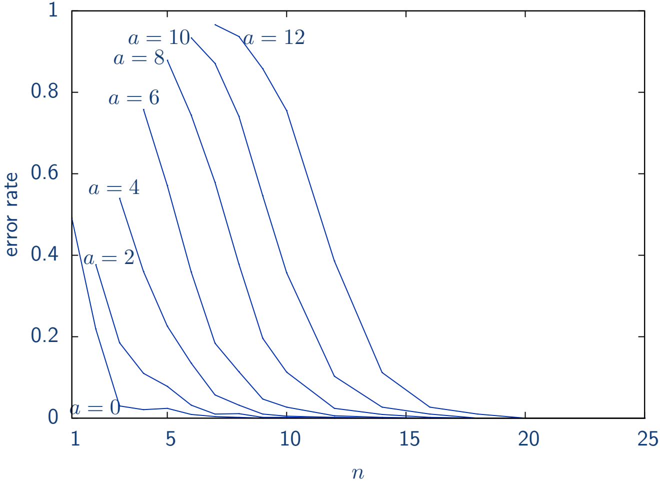
- rank $r = 2 + a$
- drawback: guess a values during decryption

Effect of Minus

- fixed $n = 35$



Decryption Errors



Differential Symmetry Attack

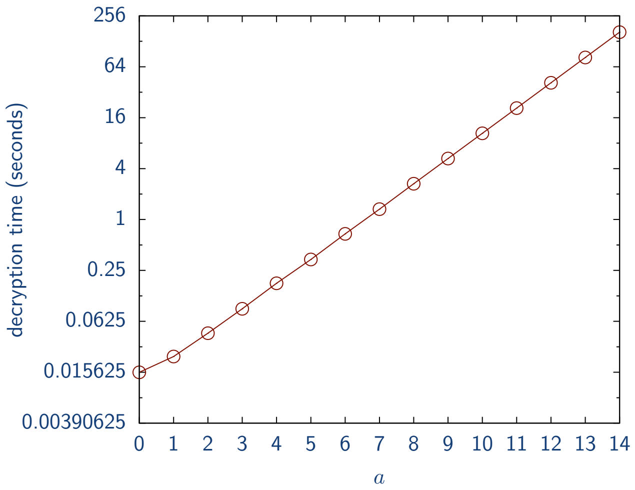
- $D\mathcal{F}(\mathbf{x}, \mathbf{y}) = \mathcal{F}(\mathbf{x} + \mathbf{y}) - \mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y}) + \mathcal{F}(\mathbf{0})$
- symmetry $\Leftrightarrow \exists\, \Lambda, L\ .\ D\mathcal{F}(L\mathbf{x}, \mathbf{y}) + D\mathcal{F}(\mathbf{x}, L\mathbf{y}) = \Lambda D\mathcal{F}(\mathbf{x}, \mathbf{y})$
- broke SFLASH
- solution (pSFLASH): S must be singular and n prime
- EFC$_p$:
    - rank$(A) =$ rank$(B) = n - 1$
    - n is prime
    - and ker$(A) \cap$ ker$(B) = \{\mathbf{0}\}$

Estimating Security

- algebraic attack: Gaussian elimination in matrix with $T = \binom{n}{D_{\text{reg}}}$ monomials
- $\tau = \binom{n}{2}$ nonzero terms per row
- complexity of Wiedemann algorithm: $O(\tau T^2)$
- $$D_{\text{reg}} \leq \frac{(q-1)(r+a)}{2} + 2$$

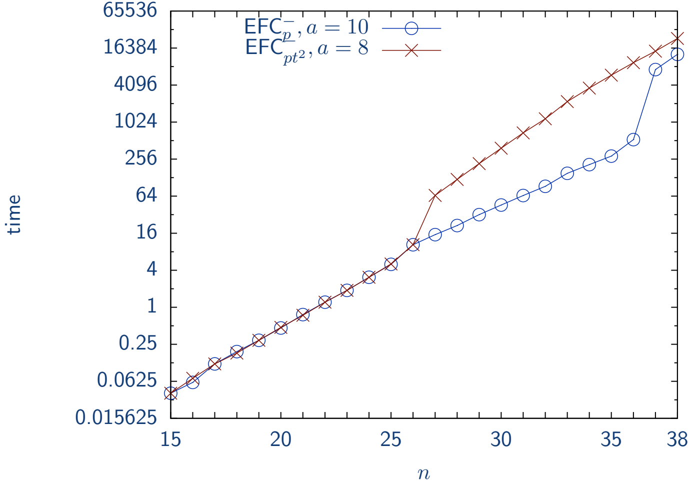
n	q	t^2	a	D_{reg}	security
83	2		10	8	82
83	2	✓	8	8	82
59	3		6	10	82

Decryption Time as a Function of a



Algebraic Attack Time

- implementation in Magma (has F_4)



Implementation Results

construction	sec. key	pub. key	ctxt.
$\mathrm{EFC}_p^-, q=2, n=83, a=10$	48.3 KB	509 KB	20 B
$\mathrm{EFC}_{pt^2}^-, q=2, n=83, a=8$	48.3 KB	523 KB	20 B
$\mathrm{EFC}_p^-, q=3, n=59, a=6$	48.8 KB	375 KB	28 B

construction	key gen.	enc.	dec.
$\mathrm{EFC}_p^-, q=2, n=83, a=10$	2.45 s	0.004 s	9.074 s
$\mathrm{EFC}_{pt^2}^-, q=2, n=83, a=8$	3.982 s	0.004 s	2.481 s
$\mathrm{EFC}_p^-, q=3, n=59, a=6$	2.938 s	0.004 s	12.359 s

Conclusion

- extension field cancellation (EFC)
  - MQ mixed field trapdoor construction
  - generate a pair of high-degree quadratic polynomials
  - uses commutativity of extension field to cancel the polynomials' complexity
  - end up with a linear system
- modifiers
  - Frobenius Tail in char 2 (speed)
  - Minus (protects against Algebraic Attack)
  - Projection (destroys Differential Symmetry)
- future work
  - get rid of Minus modifier
  - better security argument
  - shrink public keys
  - hardware implementation