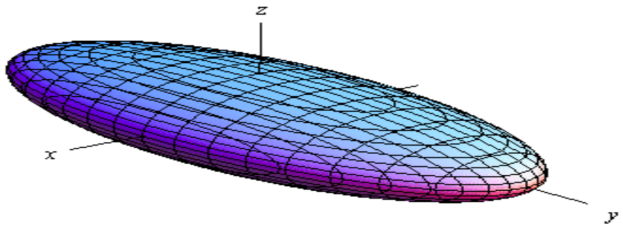
Remarks on the error distributions in ring-based LWE



Wouter Castryck[1,2], Ilia Iliashenko[1], Frederik Vercauteren[1,3]

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HEAT

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Cryptography should get ready for the post-quantum era...

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Main contenders:

- Solving multivariate quadratic systems mod p:

$$\begin{cases} s_0^2 - 3s_1s_4 + s_0s_5 + 2s_0s_3 - s_6^2 + s_1 + s_4 = 5, \\ s_0s_1 - 3s_1s_6 + 5s_4s_5 + s_5^2 + s_1 + 2s_4 - s_5 - 3s_6 = 2, \\ \vdots \end{cases}$$

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Questions:

- Can an attacker exploit the ring structure?
- What is Ring-LWE in fact (lot of confusion)?

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Currently no threat to Ring-LWE.

1. Learning With Errors (LWE)

The LWE problem (O. Regev, '05): solve a linear system

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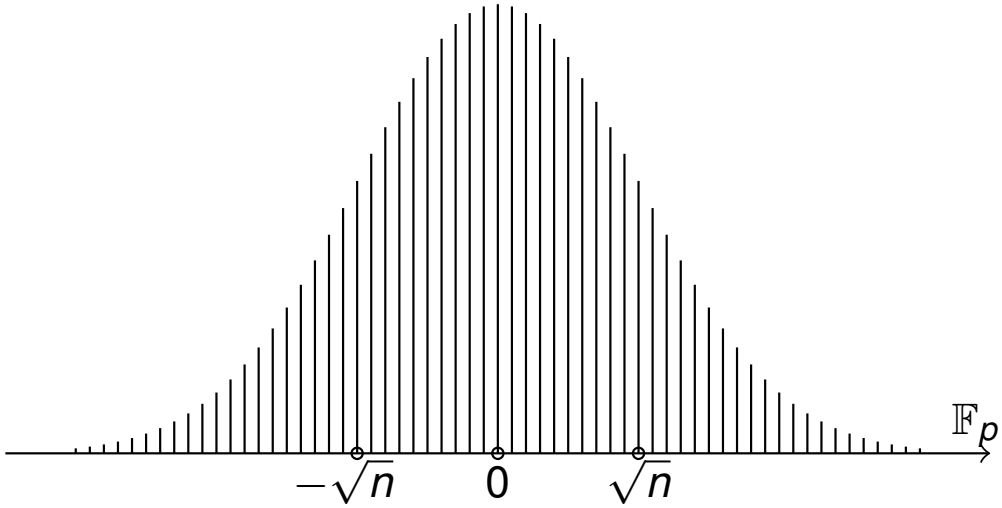
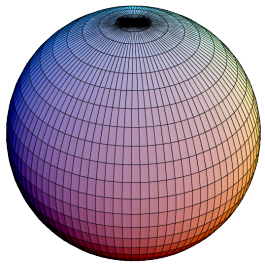
The errors e_i are sampled independently from a Gaussian with standard deviation $\sigma \gtrsim \sqrt{n}$:



When viewed jointly, the error vector

$$\begin{pmatrix} e_0 \\ \vdots \\ e_{m-1} \end{pmatrix}$$



is sampled from a spherical Gaussian.

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Known attacks for $q = \text{poly}(n)$:

- Trial and error:
  $2^{O(n \log n)}$ time and $O(n)$ samples.
- A. Blum, A. Kalai, H. Wasserman '03:
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  Idea: if all errors (almost) certainly lie in $\{-T, \ldots, T\}$, then

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  View as linear system of equations in $\approx n^{2T}$ monomials.

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LWE is tightly related to classical lattice problems.

- Can be thought of as an instance of BDD inside the lattice

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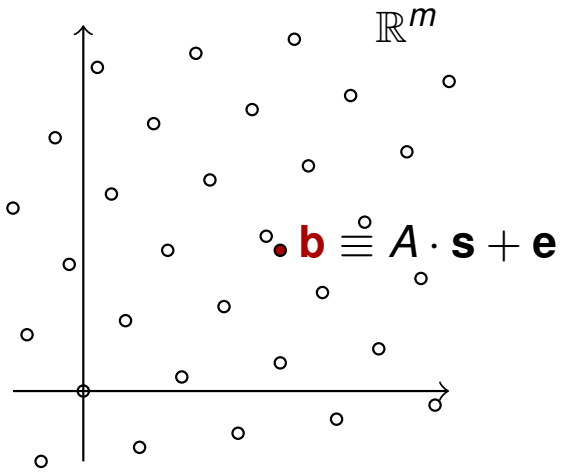

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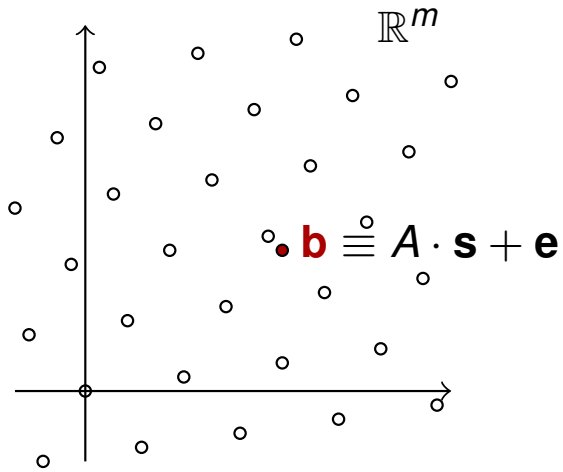

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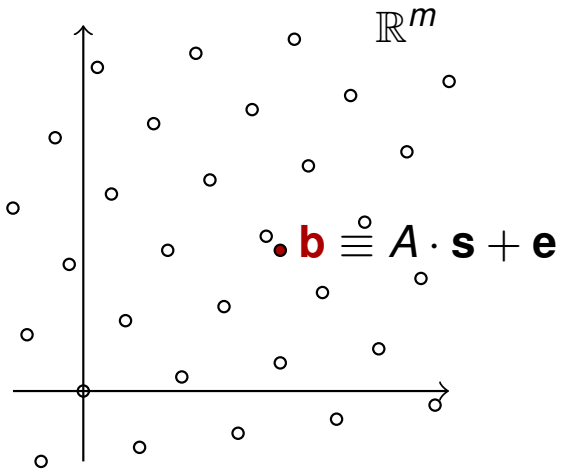

- Proven to be at least as hard as worst-case SVP-type problems, by O. Regev in ‘05 and C. Peikert in ‘09.
- Not known to be broken by quantum computers.

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Application: public-key encryption of a bit (O. Regev, '05).

- Private key: $\mathbf{s} \in \mathbb{F}_p^n$.
- Public key pair: A and $\mathbf{b} = A \cdot \mathbf{s} + \mathbf{e}$.

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- Encrypt: pick random row vector $\mathbf{r}^T \in \{0,1\}^m \subset \mathbb{F}_p^m$. Output the pair
  - $\mathbf{a}^T := \mathbf{r}^T \cdot A$ and $b := \mathbf{r}^T \cdot \mathbf{b}$ if the bit is 0,
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- Decryption of pair $\mathbf{a}^T, b$: compute

$$b - \mathbf{a}^T \cdot \mathbf{s} = b - \mathbf{r}^T \cdot A \cdot \mathbf{s} \approx b - \mathbf{r}^T \mathbf{b} = \begin{cases} 0 & \text{if bit was 0,} \\ \lfloor p/2 \rfloor & \text{if bit was 1.} \end{cases}$$

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Drawback: key size.

- To hide the secret one needs an entire linear system:

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$m \log p$ $\quad$ $mn \log p$ $\quad$ $n \log p$

2. Ring-based LWE

Solution:

- Identify key space

$$\mathbb{F}_p^n \qquad \text{with} \qquad \frac{\mathbb{Z}[x]}{(p, f(x))}$$

for some monic deg n polynomial $f(x) \in \mathbb{Z}[x]$, by viewing

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- Store $\mathbf{a}(x)$ rather than $A_{\mathbf{a}}$: saves factor n.

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Example:

- if $f(x) = x^n - 1$, then $A_{\mathbf{a}}$ is the circulant matrix

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- Bad example, because of . . .

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Potential threat:

- Suppose $f(1) \equiv 0 \bmod p$, then

$$\frac{\mathbb{Z}[x]}{(p, f(x))} \to \mathbb{F}_p : \mathbf{r}(x) \mapsto \mathbf{r}(1) = r_0 + r_1 + \cdots + r_{n-1},$$

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$$\mathbf{b}(x) = \mathbf{a}(x) \cdot \mathbf{s}(x) + \mathbf{e}(x)$$

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Safety measure: restrict to irreducible $f(x) \in \mathbb{Z}[x]$.

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Direct ring-based analogue of LWE-sample would read

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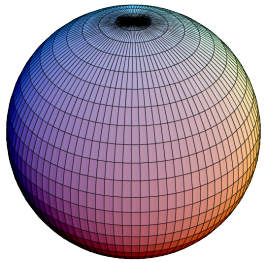

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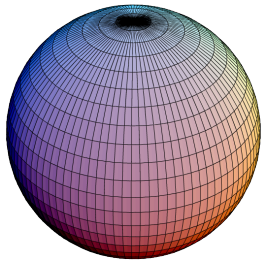

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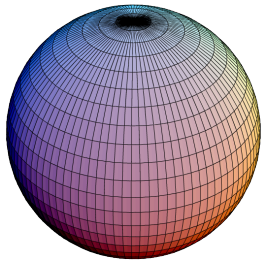

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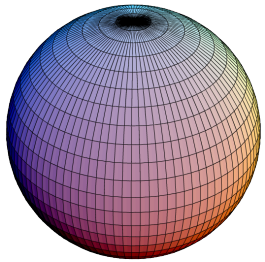

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- Sometimes called Poly-LWE.

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So what is Ring-LWE according to [LPR]? Samples look like

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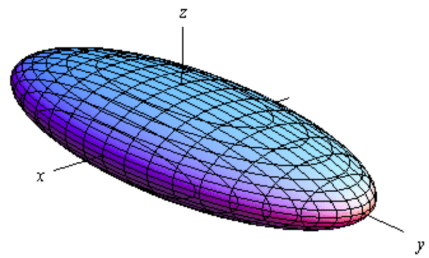

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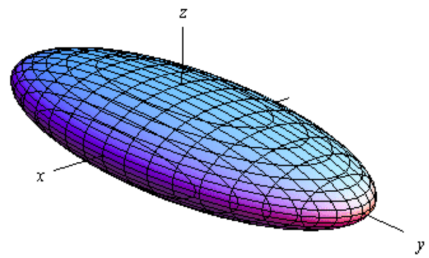

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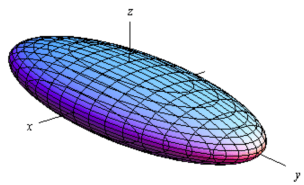
So "on average", each e_i is scaled up by $\sqrt{\Delta}^{1/n}$ . . .

- . . . but remember: skewness.



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[ELOS] constructed families of polynomials $f(x)$ that are vulnerable to an evaluation-at-1 attack.

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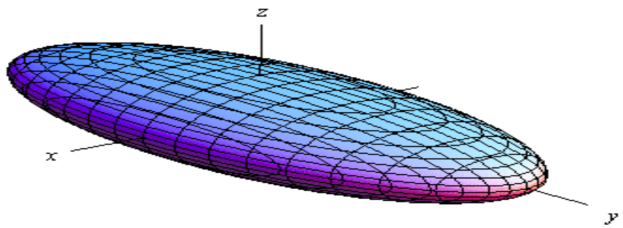
- The factor $\sqrt{\Delta}^{1/n}$ compensates for B^{-1} only "on average".
- In some coordinates B^{-1} could scale down much more.



Compensation factor is insufficient

$\rightsquigarrow$ merely rounding yields exact equations in the secret!

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All instances from [ELOS] suffer from this skewness.

- Example: $f(x) = x^{256} + 8190$, $p = 8191$. ← note: $f(1) \equiv 0 \bmod p$
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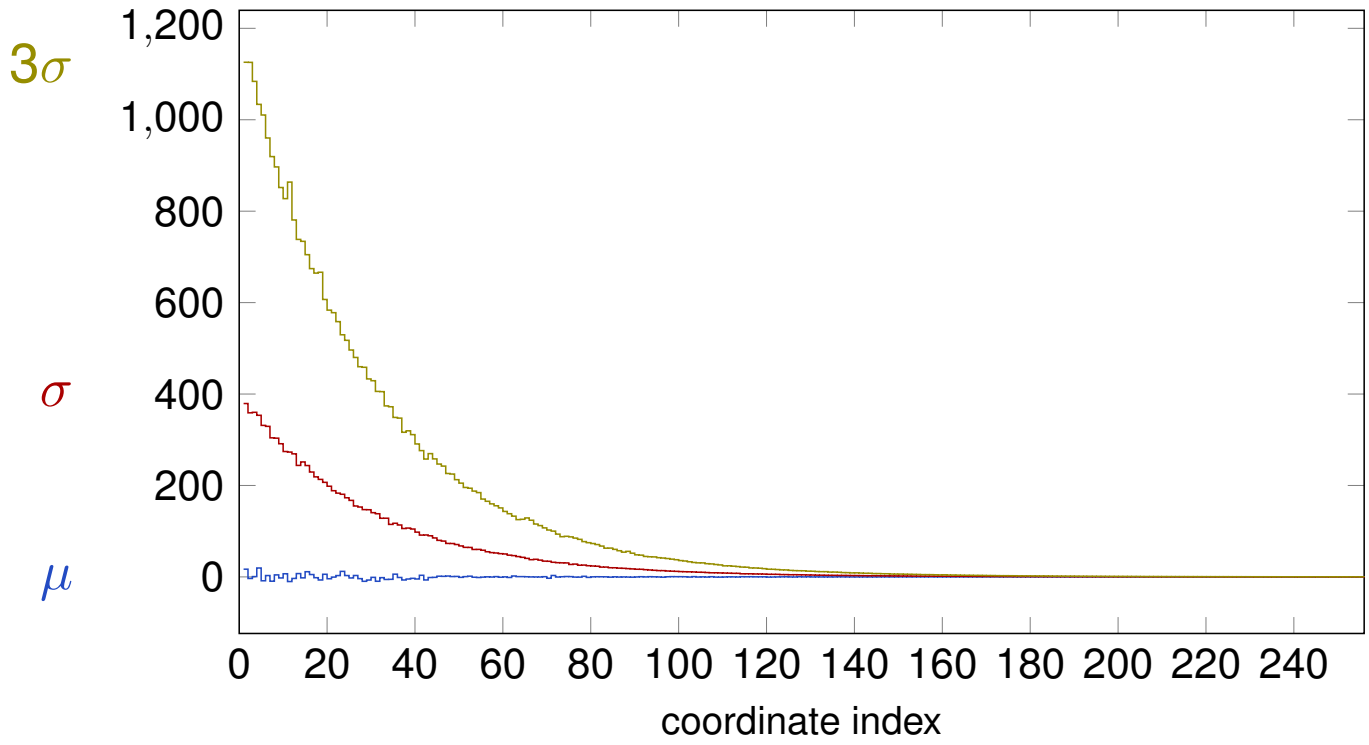

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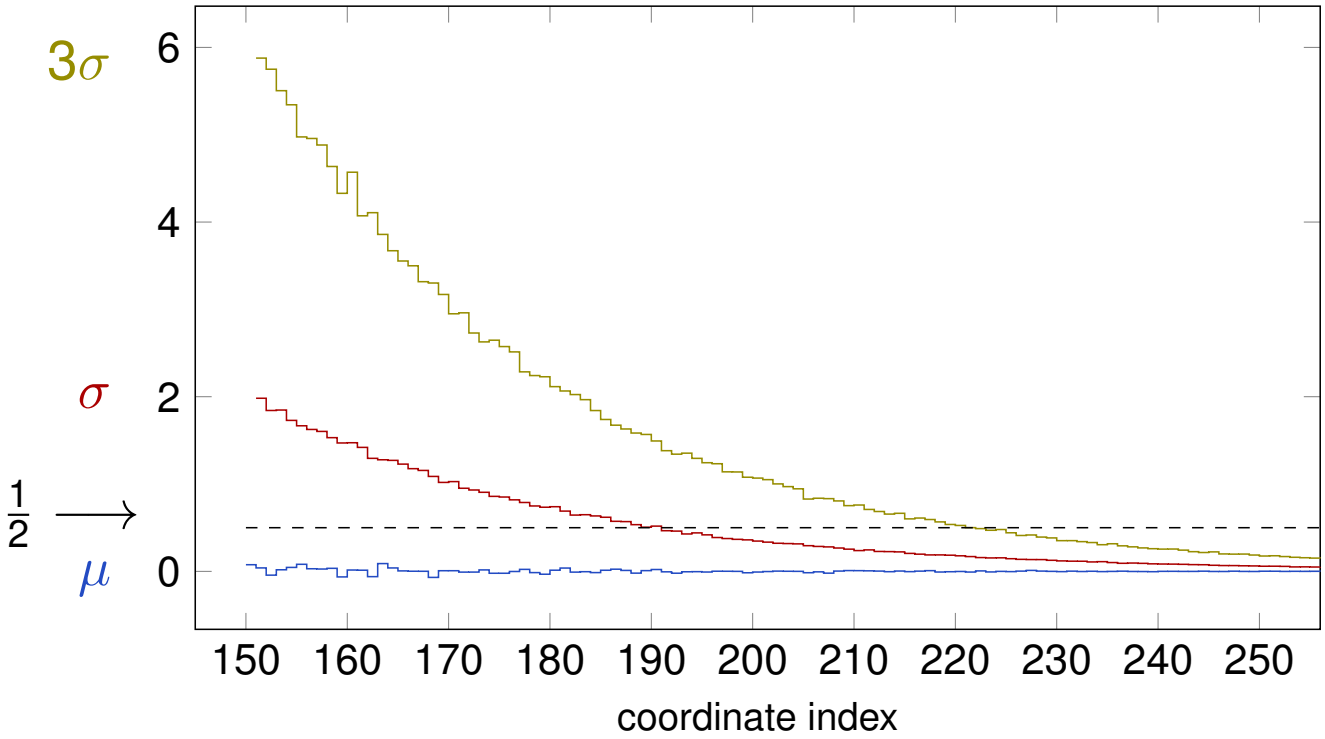

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Similar remarks apply to the other instances from [ELOS].

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- The cyclotomic case seems naturally protected against geometric growth.